Partial Pontryagin duality for actions of quantum groups on C*-algebras

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Takesaki–Takai duality

Pontryagin duality

Let G be a locally compact abelian group.
$\leftrightarrow$ another locally compact abelian group $\widehat{G}$: the Pontryagin dual of G.

Consider a C*-algebra A with a continuous action of G.
$\rightsquigarrow$ $G \ltimes A$: the reduced crossed product, a C*-algebra with an action of $\widehat{G}$,
$\rightsquigarrow$ $\widehat{G} \ltimes G \ltimes A$: with an action of $\widehat{\widehat{G}} \cong G$.

Theorem (Takesaki–Takai duality)

We have a canonical $*$-isomorphism preserving G-actions,

$$\widehat{G} \ltimes G \ltimes A \cong \mathcal{K}(L^2(G)) \otimes A.$$

When G is not abelian, $G \ltimes_{\mathrm{r}} A$ still has an "action of $\widehat{G}$". The framework of locally compact quantum groups gives an appropriate meaning to $\widehat{G}$.

Locally compact quantum group (LCQG)

Definition (Kustermans–Vaes)

A pair $G = (L^\infty(G), \Delta_G)$ of

- a von Neumann algebra $L^\infty(G)$,
- a normal unital $*$-homomorphism $\Delta_G \colon L^\infty(G) \to L^\infty(G) \bar{\otimes} L^\infty(G)$,

is a locally compact quantum group (LCQG in short here) if

(1) $(\Delta_G \otimes \mathrm{id})\Delta_G = (\mathrm{id} \otimes \Delta_G)\Delta_G$, (Coassociativity)

(2) There are faithful normal semi-finite weights ϕ, ψ that are left and right invariant, respectively. (i.e. for any normal state $\omega \in L^\infty(G)_*$, $\phi(\omega \otimes \mathrm{id})\Delta_G = \phi$ and $\psi(\mathrm{id} \otimes \omega)\Delta_G = \psi$.) (left/right Haar weights)

Associated with a LCQG G, we have:

- The C*-subalgebra $C_0(G) \subset L^\infty(G)$ of "continuous functions".
- Its Pontryagin dual $\widehat{G} = (L^\infty(\widehat{G}), \widehat{\Delta}_G)$ as a LCQG.
- Its opposite $G^{\mathrm{op}} = (L^\infty(G), \sigma\Delta_G)$ of G as a LCQG. (Here, $\sigma \colon x \otimes y \mapsto y \otimes x$ is the flip.)

Baaj–Skandalis duality

Definition

For a LCQG G and a C*-algebra A, an injective non-degenerate $*$-homomorphism $\alpha\colon A \to \mathcal{M}(C_0(G) \otimes A)$ is a (continuous) G-action on A if

(1) $(\Delta_G \otimes \mathrm{id})\alpha = (\mathrm{id} \otimes \alpha)\alpha$, (Comodule property)

(2) $\overline{\mathrm{span}}\big((C_0(G) \otimes 1)\alpha(A)\big) = C_0(G) \otimes A$. (Podleś condition)

We often assume a condition called regularity on a LCQG G.
(i.e. $\overline{\mathrm{span}}\big(C_0(G)C_0(\widehat{G})\big) = \mathcal{K}(L^2(G))$.)

Theorem (Baaj–Skandalis)

Consider a regular LCQG G and a C*-algebra A with a G^{op}-action.

- The reduced crossed product $G^{\mathrm{op}} \ltimes_{\mathrm{r}} A$ has a $\widehat{G}$-action.
- We have a canonical Morita equivalence $\widehat{G} \ltimes_{\mathrm{r}} G^{\mathrm{op}} \ltimes_{\mathrm{r}} A \overset{\mathrm{M}}{\sim} A$, preserving G^{op}-actions.

Example: Zappa–Szép product

Let G, H, D be (discrete) groups such that $G^{\mathrm{op}}, H \subset \mathrm{D}$ are subgroups with

$$G^{\mathrm{op}} \cap H = \{e\} \quad \text{and} \quad G^{\mathrm{op}} \cdot H = \mathrm{D}.$$

(Such D is called a Zappa–Szép product of G^{op} and H.)

$\rightsquigarrow$ We have maps $g\colon G \times H \to G$ and $h\colon G \times H \to H$ s.t.

$$yx = g(x,y)h(x,y) \quad \text{for} \quad x \in G,\ y \in H.$$

$\rightsquigarrow$ The map $h(x,-)\colon H \to H$ for $x \in G$ gives a G-action on the set H.

$\rightsquigarrow$ The pair $\mathrm{B} := (G \mathbin{\bar{\ltimes}} \ell^\infty(H), \Delta_{\mathrm{B}})$ is a LCQG, where for $x \in G$, $y \in H$,

$$\Delta_{\mathrm{B}}\colon G \mathbin{\bar{\ltimes}} \ell^\infty(H) \to (G \mathbin{\bar{\ltimes}} \ell^\infty(H)) \mathbin{\bar{\otimes}} (G \mathbin{\bar{\ltimes}} \ell^\infty(H)).$$

$$\ell^\infty(H) \ni \qquad \delta_y \mapsto \sum_{z \in H} \delta_z \otimes \delta_{z^{-1}y}$$

$$\mathcal{UL}(G) \ni \qquad u_x \mapsto \sum_{z \in H} u_{g(x,z)} \otimes (u_x \delta_z)$$

(Here $\delta_y(z) := \delta_{y,z}$ for Dirac delta, and u_x is the left translation of x.)

We consider a duality between actions of D and B.

Matching on locally compact quantum groups

More generally, associated to LCQGs G, H, and data m of the "commutation relation" of G^{op} and H, we have two LCQGs:

- $\mathrm{D}(\mathrm{m}) = (L^\infty(G) \mathbin{\bar{\otimes}} L^\infty(H), \Delta_{\mathrm{D}(\mathrm{m})})$, called the double crossed product by Baaj–Vaes. It contains G^{op} and H.
- $\mathrm{B}(\mathrm{m}) = (G \mathbin{\bar{\ltimes}} L^\infty(H), \Delta_{\mathrm{B}(\mathrm{m})})$, called the bicrossed product by Vaes–Vainerman. It contains $\widehat{G}$, and H is a "quotient".

More precisely, m is given as a $*$-automorphism on $L^\infty(G) \mathbin{\bar{\otimes}} L^\infty(H)$.

Definition (Vaes–Vainerman, Baaj–Vaes)

For LCQGs G and H, a $*$-automorphism $\mathrm{m} \in \mathrm{Aut}(L^\infty(G) \mathbin{\bar{\otimes}} L^\infty(H))$ is a matching on G and H if

$$(\Delta_G \otimes \mathrm{id})\mathrm{m} = \mathrm{m}_{23}\mathrm{m}_{13}(\Delta_G \otimes \mathrm{id}),$$
$$(\mathrm{id} \otimes \Delta_H)\mathrm{m} = \mathrm{m}_{13}\mathrm{m}_{12}(\mathrm{id} \otimes \Delta_H).$$

Here we used leg numbering notation, such as $\mathrm{m}_{23} = \mathrm{id} \otimes \mathrm{m}$.

Example: Quantum double

For a LCQG G, let $W^G \in L^\infty(G) \bar{\otimes} L^\infty(\widehat{G})$ be its Kac–Takesaki operator.
$\rightsquigarrow$ $\mathrm{m} := \mathrm{Ad}\, W^G \in \mathrm{Aut}(L^\infty(G) \bar{\otimes} L^\infty(\widehat{G}))$ is a matching of G^{op} and $\widehat{G}$.

The following LCQG is called the quantum double of G,

$$\begin{aligned} \mathrm{D}(G) &:= \mathrm{D}(\mathrm{Ad}\, W^G) \\ &= \Big(L^\infty(G) \bar{\otimes} L^\infty(\widehat{G}), (\mathrm{id} \otimes (\sigma\, \mathrm{Ad}\, W^G) \otimes \mathrm{id})(\Delta_G \otimes \widehat{\Delta}_G) \Big). \end{aligned}$$

Here $\sigma \colon x \otimes y \mapsto y \otimes x$ is the flip.

$\rightsquigarrow$ The bicrossed product $\mathrm{B}(\mathrm{Ad}\, W^G)$ is isomorphic to $\widehat{G} \times \widehat{G}^{\mathrm{op}}$.

Example

For a semisimple compact Lie group G and $0 < q < 1$, we have a compact quantum group G_q, a q-deformation of G.
$\rightsquigarrow$ $\mathrm{D}(G_q)$ is considered as the q-deformation of the complexification of G.

Partial version of Baaj–Skandalis duality

Theorem (K.)

For a matching m on regular LCQGs G and H, we have a one-to-one correspondence up to equivariant Morita equivalences

$$\text{(D(m)-actions on C*-algebras)} \leftrightarrow \text{(B(m)-actions on C*-algebras)},$$

under "continuity" on the matching m and the action $G \curvearrowright H$ (precisely, if $\mathrm{m}|_{C_0(G)\otimes C_0(H)} \in \mathrm{Aut}(C_0(G) \otimes C_0(H))$ and it gives a G-action on $C_0(H)$).

When H is trivial, this is same to Baaj–Skandalis duality. Also, such phenomena have been observed in many cases:

- The procedure of (LHS)→(RHS) was given by Baaj–Vaes in von Neumann algebraic case.
- For compact quantum groups G_1, G_2, weak Morita equivalence of $\mathrm{Rep}(G_1)$ and $\mathrm{Rep}(G_2)$ was investigated by Neshveyev–Yamashita.
- When $\mathrm{D(m)} = \mathrm{D}(H)$ and H is a compact group, this is due to Voigt.

Quantum group equivariant KK-theory

- Associated to a pair of separable C*-algebras A and B, there is an abelian group $KK(A, B)$, the "bivariant version" of K-groups of C*-algebras. (Kasparov)
  - We have $\mathrm{KK}(\mathbb{C}, B) = K_0(B)$.
  - We have an associative product $KK(A, B) \times KK(B, D) \to KK(A, D)$.
- When A, B have actions of a quantum group G, there is an equivariant version $KK^G(A, B)$ of KK-groups. (Baaj–Skandalis)

From now, we assume separability of $C_0(G)$ and regularity for a LCQG G.

The G-equivariant KK-theory gives rise to a category KK^G of

- Objects: separable C*-algebras with G-actions.
- Morphisms: elements of $KK^G(A, B)$ for separable C*-algebras A, B with G-actions.

$\rightsquigarrow$ The previous duality gives a categorical equivalence $\mathrm{KK}^{\mathrm{D(m)}} \simeq \mathrm{KK}^{\mathrm{B(m)}}$.

Quantum analog of the Baum–Connes conjecture

For $\mathcal{S} \subset \mathrm{KK}^G$, we put $\langle \mathcal{S} \rangle \subset \mathrm{KK}^G$ for the localizing subcategory generated by $\mathcal{S}$. (Closed under countable direct sums and mapping cones.)

Proposition

For a compact quantum group G, the following are equivalent.

(1) $\mathrm{KK}^{G \times G^{\mathrm{op}}} = \langle A \mid A \in \mathrm{KK} \rangle$.

(2) $\mathrm{KK}^{\mathrm{D}(G)} = \langle \mathrm{Ind}_G^{\mathrm{D}(G)} A \mid A \in \mathrm{KK}^G \rangle$.
(Here, $\mathrm{Ind}_G^{\mathrm{D}(G)} A$ means the induced $\mathrm{D}(G)$-action.)

- When $\widehat{G}$ is torsion-free, (1) [resp. (2)] is a (candidate of) quantum analog of the property of "$\gamma = 1$" for $\widehat{G} \times \widehat{G}^{\mathrm{op}}$ [resp. $\mathrm{D}(G)$] via the reformulation of the Baum–Connes conjecture by Meyer–Nest.
- The property of $\gamma = 1$ holds for amenable (or a-T-menable) groups. (Higson–Kasparov)
- For a semisimple compact Lie group G of rank≥ 2 and $0 < q < 1$, $\mathrm{D}(G_q)$ has property (T) (whereas $\widehat{G}_q \times \widehat{G}_q^{\mathrm{op}}$ is amenable). (Arano)

K-theory of (quantum) homogeneous spaces

We consider $\mathrm{D}(G)$-equivariant KK-theory of quantum homogeneous spaces.

We write $\overset{KK^G}{\sim}$ for an isomorphism in the category KK^G.

Theorem (A. Wasserman, Rosenberg–Schochet)

Let G be a connected compact Lie group with torsion-free $\pi_1(G)$ and $T \leq G$ be a maximal torus. Then $C(G/T) \overset{KK^G}{\sim} \mathbb{C}^{\oplus n}$ for some $n \in \mathbb{Z}_{>0}$.

Theorem (Voigt)

For $q \in (-1, 1) \setminus \{0\}$, it holds $C(SU_q(2)/T) \overset{KK^{\mathrm{D}(SU_q(2))}}{\sim} \mathbb{C}^{\oplus 2}$.

$\rightsquigarrow$

- $\widehat{G}$ satisfies analog of the Baum–Connes conjecture. (Meyer–Nest)
- $\widehat{SU_q(2)}$ satisfies analog of the Baum–Connes conjecture. (Voigt)

Graded twisting in the sense of Bichon–Neshveyev–Yamashita

We give a similar computation for a quantum homogeneous space of a graded twisting of compact Lie groups. Let

- G be a connected, simply connected compact Lie group,
- $0 < q \leq 1$, and
- $\tau \in \mathcal{Z}(G)^r$, where r is the rank of G.

Then Neshveyev–Yamashita constructed a graded twisting G_q^τ of G_q as a new compact quantum group.

Theorem (Neshveyev–Yamashita)

C*-tensor categories of the form $\mathrm{Rep}(SU_q(n)^\tau)$ exhausts all rigid C*-tensor categories with the fusion rule of $\mathrm{Rep}(SU(n))$, which were classified by Kazhdan–Wenzl, Jordans.

Applications to quantum homogeneous spaces

We can compare $\mathrm{KK}^{\mathrm{D}(G_q)}$ and $\mathrm{KK}^{\mathrm{D}(G_q^\tau)}$ to some extent. When $q = 1$, we get the following.

Theorem (K.)

Let G be a connected, simply connected compact Lie group, $T \leq G$ be a maximal torus, and $\tau \in \mathcal{Z}(G)^r$. Then $C(G^\tau/T) \overset{KK^{\mathrm{D}(G^\tau)}}{\sim} \mathbb{C}^{\oplus n}$ for some n.

Strategy

With the aid of $\mathrm{KK}^{\mathrm{D}(G^\tau)} \simeq \mathrm{KK}^{G^\tau \times G^{\tau\,\mathrm{op}}}$, we reduce the followings to the known case of $G \rtimes (T/\mathcal{Z}(G))$ for $\mathrm{Ad}\colon T/\mathcal{Z}(G) \to \mathrm{Aut}(G)$.

(1) We show $\mathrm{KK}^{\mathrm{D}(G^\tau)} = \langle \mathrm{Ind}_{G^\tau}^{\mathrm{D}(G^\tau)} A \,|\, A \in \mathrm{KK}^{G^\tau} \rangle$.

(2) Construct $\mathbf{x} \in \mathrm{KK}^{\mathrm{D}(G^\tau)}(C(G^\tau/T), \mathbb{C}^{\oplus n})$ and show $\mathbf{x}$ is a KK^{G^τ}-equivalence. By (1), $\mathbf{x}$ is also a $\mathrm{KK}^{\mathrm{D}(G^\tau)}$-equivalence.

Thank you very much for listening!